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Controller design and reduction of bullwhip for a model supply chain system using z-transform analysis

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Abstract

In this work, a discrete time series model of a supply chain system is derived using material balances and information flow. Transfer functions for each unit in the supply chain are obtained by *z*-transform. The entire chain can be modeled by combining these transfer functions into a close loop transfer function for the network. The model proves to be very useful in revealing the dynamics characteristic of the system. The system can be viewed as a linear discrete system with lead time and operating constraints. The stability of the system can be analyzed using the characteristic equation. Controllers are designed using frequency analysis. The bullwhip effect, i.e. magnification of amplitudes of demand perturbations from the tail to upstream levels of the supply chain, is a very important phenomenon for supply chain systems. We proved that intuitive operation of a supply chain system with demand forecasting will cause bullwhip. Moreover, lead time alone would not cause bullwhip. It does so only when accompanied by demand forecasting. Furthermore, we show that by implementing a proportional intergral or a cascade inventory position control and properly synthesizing the controller parameters, we can effectively suppress the bullwhip effect. Moreover, the cascade control structure is superior in meeting customer demand due to its better tracking of long term trends of customer demand.

Keywords: Supply chain; *z*-transform; Bullwhip effect; Frequency analysis; PI control; Cascade control

1. Introduction

Supply chain management has attracted much attention among process system engineering researchers recently. There are many aspects in supply chain research. One area is the analysis of the logistic problem of a supply chain using system control theory. While a real supply chain is a very complex network, we choose to focus on the material balance and information flow of the system. Hence, it is possible to derive first principle models which describe the dynamics of a supply chain system. Such models can be used as the basis for understanding supply chain dynamics. Intuitively, the management of such a system is to maintain the inventory level of each unit to satisfy the demands from its customers by ordering products from its upper stream of the supply chain. The ordering policies can be viewed as a control strategy of its inventory level. The other need of a supply chain system is to learn the change in the market, i.e. to forecast the change of the demand from the orders of the downstreams. The objective of this work is to obtain a close form solution of the dynamic model of a supply chain system using *z*-transform and analyze the ordering strategy using controller design principle.

A model of a supply chain was developed in as early as the 1960s [1]. A review on modeling and analysis of the supply chain system was provided by Beamon [2]. For example, Porter and coworkers [3–5] analyzed the feedback control of a supply chain using discrete time series simulation and D-partition analysis. Recently, Perea-López et al. [6,7] examined the dynamic behavior of a supply chain system and analyzed the impact of several heuristic control laws, using continuous time

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domain simulations. For a continuous time domain model, it is customary to conduct a theoretical analysis using the Laplace transform to convert ordinary differential equations into s-domain transfer functions. For a system with discrete sampling time, the z-transform is used to convert the time series model into z-domain transfer function which can be obtained. These close form transfer functions allow us to gain an insight into the stability of the system. Many guidelines for controller designs developed using transfer functions and frequency analysis can then be applied to devise the ordering strategy.

A useful model of a supply chain must be able to reproduce the most important dynamic characteristics of a supply chain system, the "bullwhip effect". Small perturbations in the customer demand of the downstream units will cause huge perturbations of the orders to upstream units [8,9]. The bullwhip is usually attributed to the lead time between ordering and delivery, aggressive ordering, demand forecasting [10,11]. Towill and coworkers [12,13] examined the role of demand forecasting in such systems using the z-transform transfer function analysis. Recently, they have analyzed the effect of the ordering strategy [14]. Chen et al. [11] discussed the merit of using an exponential filter in forecasting, also in a feed-forward context.

In this work, a supply chain model is analyzed using the z-transform. Analytical forms of the closed loop transfer functions are obtained. The causes of bullwhip become quite apparent using the model and stability analysis. A PI and cascade control structures are proposed and controllers are synthesized and tuned accordingly, to eliminate the bullwhip effect. In Section 2 the basic supply chain model and its z-transform will be introduced.

2. Basic dynamic model

Consider a simple supply chain that has no branch as shown in Fig. 1. There are three logistic echelons: warehouse (W), distributing center (D) and retailer (R) between the producer (P) and customer (C). Let $I_j(t)$ denote the inventory of each logistic node at any time instant t, where $j \in \{W, D, R\}$. The amount of goods delivered to node j by the upstream node i is denoted by $Y_{ij}(t)$, $ij \in \{PW, WD, DR, RC\}$. A time delay of L is assumed for all delivery actions so that goods dispatched at time t will arrive at time $t+L$. However, due to the need for examination and administrative processing, this new delivery is only available to a customer at $t+L+1$. The inventory balance at node j is given by:

$$I_j(t) = I_j(t-1) + Y_{ij}(t-L) - Y_{jk}(t), \quad jk \in \{WD, DR, RC\} \tag{1}$$

Due to the delay in delivery, an inventory position $\mathrm{IP}_j(t)$, $j \in \{W, D, R\}$ is defined to better monitor the change in the inventory:

$$\mathrm{IP}_j(t) = \mathrm{IP}_j(t-1) + Y_{ij}(t) - Y_{jk}(t) \tag{2}$$

In a decentralized supply chain, the manager aims at maintaining a certain inventory position. The amount of orders placed by node j to an upstream node i is denoted by $U_{ji}(t)$, $ji \in \{WP, DW, RD, CR\}$. For example, simple P-control can be used:

$$U_{ji}(t) = K_j \times (\mathrm{SP}_j(t) - \mathrm{IP}_j(t)) \tag{3}$$

where SP_j is the set point of the inventory position at node j. Note that the intuitive setting of this controller is $K_j = 1$. The advantages of using the inventory position instead of actual inventory will be evident later in our analysis. It should be pointed out that the size of the order placed corresponds to a controller decision. A controller output has no constraints. We allow the downstream customer node to order as much as it wants, with no guarantee that the order can be fulfilled. Similarly, we allow the downstream node to retract its order so that $U_{kj}(t)$ can be negative if the inventory position is higher than the set point.

We assume that ordering information is communicated instantaneously. However, an order at time t will only be processed at time $t+1$, due to administrative delay. Therefore, a standing order for each node j at time t, $O_j(t)$, $j \in \{W, D, R\}$ is defined as the amount of order to be processed at time $t+1$. Moreover, we assume that an order can be accumulated to the next time step if it is not fulfilled, since each customer has only one supplier in our simple supply chain. Therefore, the standing order for node j at time t is the sum of the order placed plus any unfulfilled order at time t:

$$O_j(t) = U_{kj}(t) + O_j(t-1) - Y_{jk}(t) \tag{4}$$

The actual delivery, corresponding to a control valve's action, has physical limits. If there is enough inventory to satisfy the standing order at $t-1$, all the orders will be delivered. Otherwise, the inventory will be cleared

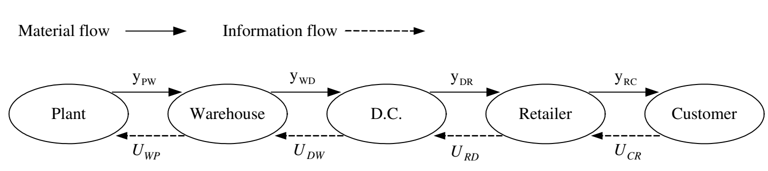

Fig. 1. A simple supply chain.

(i.e. the valve is fully open). Similarly, if the downstream node already has too much inventory, the supplier will just stop delivery (i.e. the valve is fully closed); return of goods is not taken into consideration. Therefore

$$Y_{jk}(t) = \begin{cases} 0 & O_j(t-1) \leqslant 0 \\ O_j(t-1) & 0 \leqslant O_j(t-1) \leqslant I_j(t-1) \\ I_j(t-1) & 0 \leqslant I_j(t-1) \leqslant O_j(t-1) \end{cases} \quad (5)$$

For simplicity, let the customer satisfaction of node j be represented by a back-order defined as the difference between the total standing order at $t-1$, $O_j(t-1)$ and the amount of goods actually delivered $Y_{jk}(t)$ at t:

$$\mathrm{BO}_j(t) = O_j(t-1) - Y_{jk}(t) \quad (6)$$

The larger the BO_j the poorer is the customer satisfaction.

The z-transform of the above discrete time model is given by

$$I_j(z) = \frac{z}{z-1}(z^{-L}Y_{ij}(z) - Y_{jk}(z)) \quad (7)$$

$$\mathrm{IP}_j(z) = \frac{z}{z-1}(Y_{ij}(z) - Y_{jk}(z)) \quad (8)$$

$$O_j(z) = \frac{z}{z-1}(U_{kj}(z) - Y_{jk}(z)) \quad (9)$$

$$U_{ji}(z) = K_j(\mathrm{SP}_j(z) - \mathrm{IP}_j(z)) \quad (10)$$

$$Y_{jk}(z) = \begin{cases} 0 & z^{-1}O_j(z) \leqslant 0 \\ z^{-1}O_j(z) & 0 \leqslant z^{-1}O_j(z) \leqslant z^{-1}I_j(z) \\ z^{-1}I_j(z) & 0 \leqslant z^{-1}I_j(z) \leqslant z^{-1}O_j(z) \end{cases} \quad (11)$$

The corresponding simplified block diagram is given in Fig. 2.

The above model seems simple. Nevertheless, it captures the basic dynamic feature of a supply chain system. A real supply chain usually has many customers, suppliers, and products. In a decentralized system, the inventory dynamics does not really depend on how many customers the node has, since all customer demands can be lumped into an aggregate demand. Obviously, if every node has sufficient inventory and has the same transportation delay, the distribution of order would not affect the system dynamic behavior. We can assume that different suppliers have different inventory levels and different transportation delays, and investigate the optimal order allocations. If the processing of order and delivery of different products do not interfere with each other, each product can be viewed as a separate supply chain. One may impose constraints that total inventory is limited by storage space and devise an order strategy accordingly. Various complications can be introduced and analyzed using our basic model. However, it is important to understand the basic dynamic behavior, before such complications are introduced.

3. Stability analysis

The objective of this section is to examine some asymptotic cases of the supply chain operations with a proportional control of inventory levels. The dynamic behavior may or may not involve the transition from one asymptotic regime to another depending on how large and how abrupt the change in the customer demand is. However, such asymptotic analysis provides extremely useful insights.

3.1. Infinite supply and high stock

In this case, we assume that the upstream supplier has sufficient inventory so that the demand of node j is always satisfied: i.e. $Y_{ij}(z) = z^{-1}U_{ji}(z)$. Furthermore, we assume that the set point of node j is sufficiently high so that there will always be sufficient inventory to satisfy all customer demands, i.e. $Y_{jk}(z) = z^{-1}O_j(z) = z^{-1}U_{kj}(z)$. The closed loop transfer function can be derived as the following equation (see Appendix A.1):

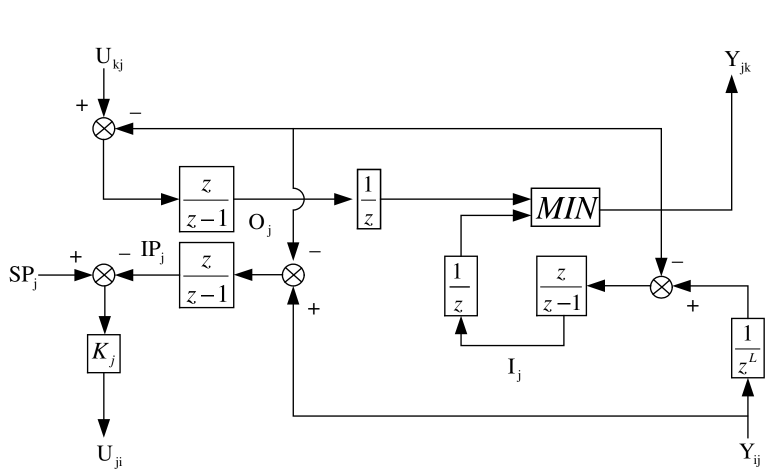

Fig. 2. The block diagram of node j of a supply chain.

$$\mathrm{IP}_j(z) = \frac{K_j \times \mathrm{SP}_j(z) - U_{kj}(z)}{z - 1 + K_j} \tag{12}$$

with a characteristic equation:

$$H_j(z) = z + K_j - 1 = 0 \tag{13}$$

A sampled data system is stable if all the roots of the characteristic equation lie within the unit circle:

$$H_j(z^*) = z^* + K_j - 1 = 0 \Rightarrow |z^*| = |K_j - 1| \leqslant 1$$
$$\Rightarrow 0 \leqslant K_j \leqslant 2 \Rightarrow K_{j,U} = 2$$

Therefore, if upstream supply is infinite and the inventory position set point is sufficiently high that a bang–bang situation is never reached unless the system becomes unstable, then the ultimate gain of the feedback loop $K_{j,U}$ is equal to 2. It is interesting to note that the intuitive setting of this controller is $K_j = 1$, which corresponds exactly to the Ziegler–Nichols quarter-decay tuning results for proportional only control.

3.2. Infinite supply and low stock

If an upstream supplier has sufficient inventory so that the demand of node j is always satisfied, i.e. $Y_{ij}(z) = z^{-1}U_{ji}(z)$, but the set point of node j is low so that there will always be insufficient inventory to satisfy all customer demands, i.e. $Y_{jk}(z) = z^{-1}I_j(z)$, then, the following closed loop transfer function and characteristic equation are obtained (see Appendix A.2):

$$\mathrm{IP}_j(z) = \frac{\frac{K_j(z^{L+1}-1)}{z-1}}{z^{L+1} + \frac{K_j(z^{L+1}-1)}{z-1}} \mathrm{SP}_j(z) \tag{14}$$

$$H_j(z) = z^{L+1} + \frac{K_j(z^{L+1} - 1)}{z - 1} = 0 \tag{15}$$

It can be shown that whenever $K_j \geqslant 1$, there exists at least one $|z^*| \geqslant 1$ (see Appendix A.3). Therefore, if the upstream supply is infinite and the inventory position set point is so low that there is always less inventory than the standing order, the sufficient condition for the unstable feedback loop is $K_j > 1$. Moreover, the closed loop transfer function (Eq. (14)) is independent of U_{kj}. Therefore, when there is unlimited upstream supply but a low stock target, the inventory position becomes independent of fluctuations in downstream demands.

3.3. Limited supply

In this case, we assume that the upstream supplier i does not have sufficient inventory so that supply to node j is not dependent on the demand of node j but limited by the availability of the inventory: i.e. $Y_{ij}(z) = z^{-1}I_i(z)$. In this case, we found the following closed loop transfer function (see Appendix A.4):

$$\mathrm{IP}_j(z) = \begin{cases} \frac{1}{z-1}(I_i(z) - U_{jk}(z)) & I_j(z) \geqslant z^{-1}U_{kj}(z) \\ \frac{z^{L+1}-1}{(z-1)z^{L+1}} I_i(z) & \text{otherwise} \end{cases} \tag{16}$$

In Eq. (16), the inventory position of node j depends neither on the set point nor on the controller gain of the ordering policy of node j. This result is intuitive. If the supplier is low in stock, no matter how node j orders, its inventory position is determined by the stock available to the supplier.

In Section 4, the "bullwhip" effect is analyzed using the transfer functions.

4. Bullwhip effect

The bullwhip effect can be represented as an amplification of demand fluctuations from downstream to upstream. In the above section, we have seen that coupling between two nodes through the ordering policy is eliminated when there is insufficient stock in any one node along the chain. Therefore, propagation of demand fluctuations is only possible when every node has sufficient stock. When there is sufficient supply and high stock, substituting the P-control equation, Eq. (10), into the transfer function (Eq. (12)) we get

$$U_{ji}(z) = \frac{K_j \times (z - 1)}{z - 1 + K_j} \mathrm{SP}_j(z) + \frac{K_j}{z - 1 + K_j} U_{kj}(z) \tag{17}$$

4.1. Aggressive ordering

One factor that "bullwhip" is usually attributed to is aggressive ordering. We have demonstrated that the system would become unstable when K_j is set greater than $K_{j,U} = 2$, the ultimate gain. Here, we show that when there is no change in the inventory position set point, the "bullwhip" effect is found only if the controller gain K_j is set greater than 1.

Assuming that there is no change in the set point, the ratio of orders to successive nodes can be expressed as

$$\frac{|U_{ji}(z)|}{|U_{kj}(z)|} = \frac{K_j}{|z - 1 + K_j|} \tag{18}$$

The amplitude in demand fluctuations will be amplified if

$$\frac{|U_{ji}(z)|}{|U_{kj}(z)|} = \frac{K_j}{|\mathrm{e}^{\mathrm{i}\omega} - 1 + K_j|} > 1 \quad \forall \omega \tag{19}$$

Further mathematical manipulations will show that the condition is met only if $K_j > 1$ (see Appendix A.5).

Fig. 3 is a plot of the magnitude ratio of Eq. (19) at various values of controller gain ($K_j = 0.9, 1.0, 1.1$). It shows that bullwhip is mainly caused by high frequency fluctuations in customer demand when $K_j > 1$, i.e. the manager of the distributing node responded too ag-

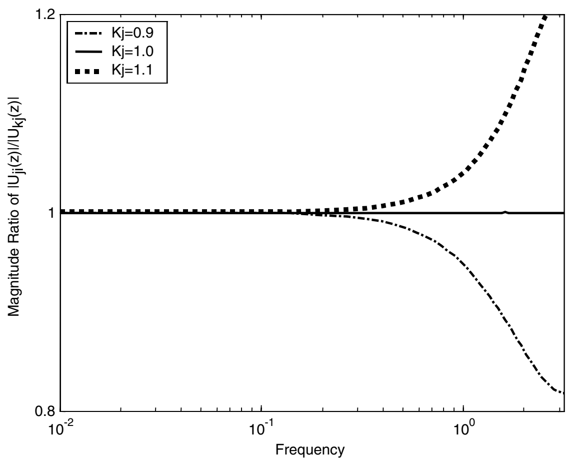

Fig. 3. The magnitude ratio vs. frequency for various values of the controller gain.

gressively to short-term fluctuations. If $K_j < 1$, the magnitude ratio can actually be reduced along the chain.

One might ask, what is the incentive for managers to order less than what is required to achieve the inventory target? If the inventory target is large enough that the safety inventory is able to cover all short-term fluctuations in demand, e.g. safety inventory equals one day of demand, or three times the variance in demand. Ordering slightly less than what is required may reduce inventory without actually sacrificing customer satisfaction. Therefore, the managers of the distributing node will implement less aggressive ordering if they are confident that the inventory position target is good enough to meet long term trends in demand.

4.2. Demand forecasting

If we attempt to forecast the customer demand and set the inventory position target accordingly, as shown in Fig. 4, the closed loop responses of inventory position IP_j and order to supplier U_{ji} become

$$\mathrm{IP}_j(z) = -\frac{1 - K_j \times F(z) \times (L+2)}{z - 1 + K_j} U_{kj}(z) \tag{20}$$

$$U_{ji}(z) = \frac{K_j \times ((L+2) \times F(z) \times (z-1) + 1)}{z - 1 + K_j} U_{kj}(z) \tag{21}$$

$F(z)$ is the forecaster used to predict the current demand. Towill and coworkers [12,13] have demonstrated that any forecaster will lead inevitably to bullwhip. For example, "backlog" is a simple policy that just passes along the last customer demand as the next inventory target:

$$F(z) = \frac{1}{z},$$

$$U_{ji}(z) = \frac{K_j}{z - 1 + K_j}\left(1 + (L+2) \times \frac{z-1}{z}\right) U_{kj}(z) \tag{22}$$

In Fig. 5 is a Bode plot of the magnitude ratio $|U_{ji}(z)|/|U_{kj}(z)|$ with $K_j = 0.7$. It can be seen that the bullwhip occurs at all frequencies even though a less than aggressive ordering policy is adopted.

Chen et al. [10] suggested the use of an exponential filter:

$$F(z) = \frac{\alpha}{z + \alpha - 1} \tag{23}$$

Fig. 5 also contains a Bode plot of the magnitude ratio $|U_{ji}(z)|/|U_{kj}(z)|$ using the exponential filter with $\alpha = 0.1$, and $K_j = 0.7$. The bullwhip can be suppressed for most frequencies.

4.3. Effect of lead time

Eq. (22) indicate that the lead time is a factor for the bullwhip effect if the supply chain takes demand forecasting into consideration of its ordering policy. Fig. 6 demonstrates that a longer lead time will lead to a stronger bullwhip if one implements *P*-only control algorithm and exponential filter with $\alpha = 0.1$, and $K_j = 0.7$. However, without demand forecasting, Eq. (18) indicates that the lead time is not a factor for the bullwhip effect.

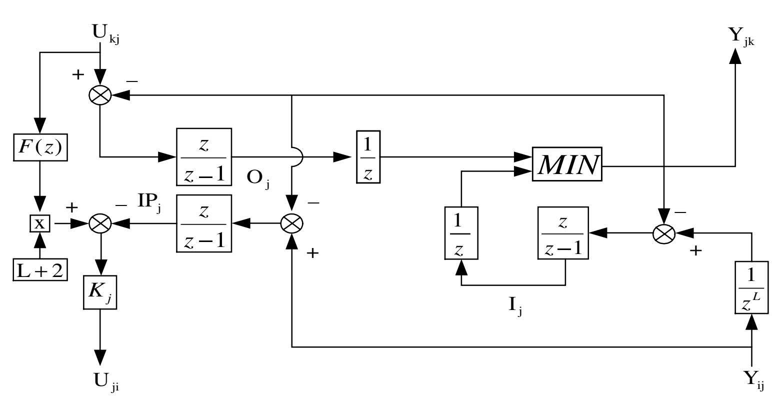

Fig. 4. The block diagram of node j of a supply chain with demand forecasting.

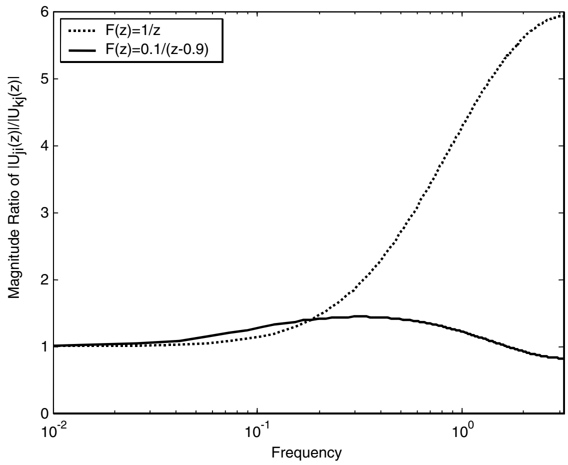

Fig. 5. The closed loop frequency responses of $|U_{ji}(z)|/|U_{kj}(z)|$ in the case of demand forecasting.

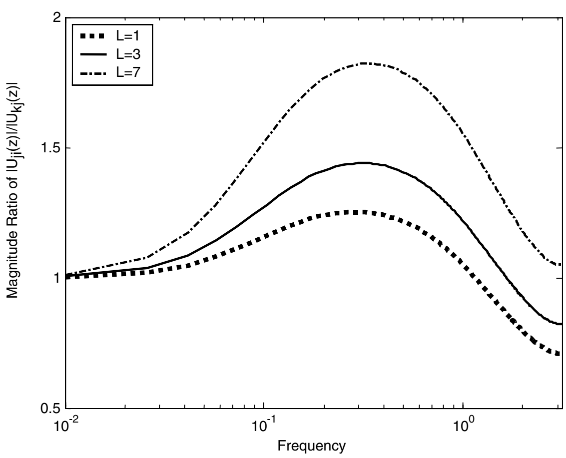

Fig. 6. Frequency responses of $|U_{ji}(z)|/U_{kj}(z)|$ in the case of various lead time values with a P-only controller at $K_j = 0.7$.

In Section 5, we will show that how a better ordering strategy can be synthesized using principles of controller design.

5. Controller synthesis

5.1. Controller tuning criterion

In this section, we assume that the customer demand is stochastic, i.e. $d \in N(m, \sigma)$. However, the average of demand may be subjected to a low frequency disturbance such as a step change or seasonal cyclic changes. The objective of a simple inventory level controller is to maintain a given inventory position in the presence of such a low frequency disturbance. However, in addition to achieving the inventory position target, the objectives of a supply chain manager also include setting an inventory position target that is not too high (resulting in excess inventory costs) or too low (resulting in customer dissatisfaction due to back-order) compared to the current average demand. Therefore, a manager should aim to create a fast response of the order to low frequency demand changes so that the inventory level can be maintained, but limit the ratio of order to demand to less than 1 at high frequency. The frequency response of $|U_{ji}(z)|/|U_{kj}(z)|$ of a closed loop supply chain node should be used for controller design. Standard textbooks [15] suggest the following two factors to be considered:

1. *Bandwidth*: the frequency at which the magnitude ratio is reduced to below 0.7. A wide bandwidth indicates a faster response but poorer noise rejection capabilities.
2. *Resonance peak (RP)*: the highest value of the amplitude ratio. A higher resonance peak indicates a faster response but may be more oscillatory. Suitable setting of RP ranges from 1.5 to 2.0.

Note that we only discuss a discrete system; therefore, the highest frequency is at $\omega = \pi$. Therefore, we can define a term MR_π as the magnitude ratio at $\omega = \pi$:

$$\mathrm{MR}_\pi = |U_{ji}(\omega)|/|U_{kj}(\omega)|\|_{\omega=\pi} \quad (24)$$

Since a higher MR implies a wider bandwidth and a faster response, it results in more severe bullwhip. Therefore the following approximate tuning criterion can be used:

"Choose a controller setting with MR_π in the range of 0.8 to 1; and RP in the range 1.5 to 2".

5.2. Simple feedback with demand forecast

Fig. 7 gives the frequency response of a unit with a proportional only controller to its inventory position

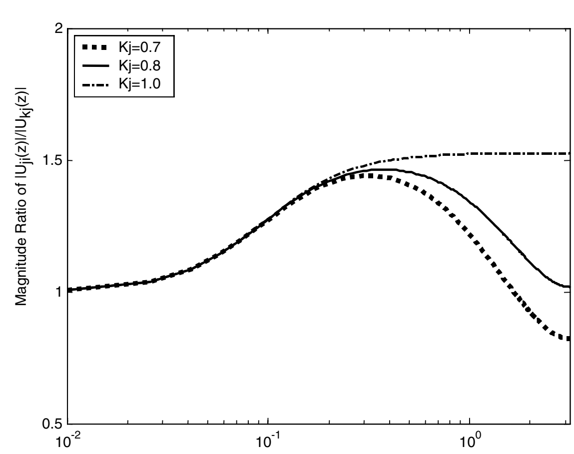

Fig. 7. Frequency responses of $|U_{ji}(z)|/|U_{kj}(z)|$ with different K_j values of the P-only controller.

and an exponential filter ($\alpha = 0.1$) with several different controller gains. It can be shown that with a controller gain lower than one, the bullwhip effect of the unit is suppressed, and $K_j = 0.7$ should be very appropriate according to the rule in the previous section. Fig. 8 shows that when $K_j = 0.7$, the bullwhip can be attenuated, while a high gain of $K_j = 1.0$ will lead to a substantial bullwhip. The responses of U_{ji} and IP_j to a step change in customer demand U_{kj} according to Eqs. (20) and (21) with a proportional gain $K_j = 0.7$ are shown in Fig. 9. The demand is originally stochastic with $d \in N(20, 4)$, but the average of the demand is subjected to a step increase to $d \in N(40, 4)$ at $t = 20$ and a step decrease to $d \in N(30, 4)$ at $t = 60$. There is a large offset between the inventory position and set point. This offset will lead to accumulation of a large amount of backorder and low customer satisfaction.

Since an offset cannot be avoided, customer dissatisfaction is inevitable for a P-only controller. To avoid this offset, a PI controller can be used:

$$C_j(z) = K_j \times \left(1 + \frac{1}{\tau_{I,j}} \frac{z}{z-1}\right) \tag{25}$$

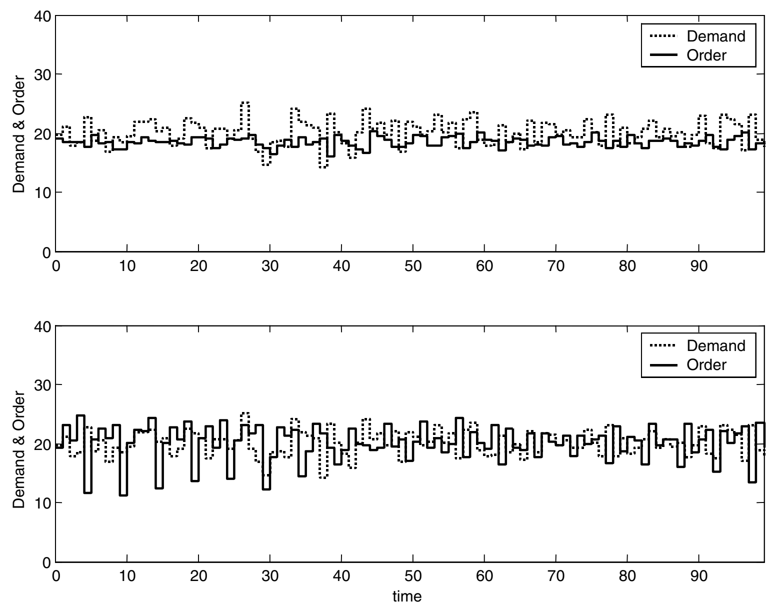

Fig. 8. Comparison of the demand and order of a unit in a supply chain using P-only controllers with different gains.

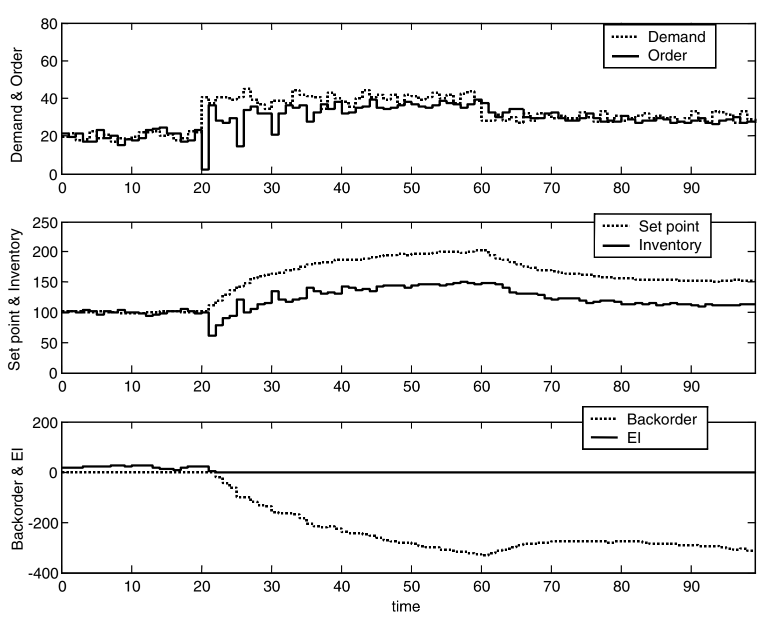

Fig. 9. Simulation results of a supply chain unit with a P-only controller ($K_j = 0.7$) and stochastic demand from downstream.

The closed loop Bode plot of one supply chain unit with a PI controller is shown in Fig. 10. Fig. 10 shows that given $\tau_{I,j} = 3.3$, the bullwhip effect still appears even the controller gain, K_j is less than one. Fig. 11 gives the contour plot of both RP and MR_π as a function of K_j and $\tau_{I,j}$. A setting of $K_j = 0.67$ and $\tau_{I,j} = 3.3$ will give an $\mathrm{MR}_\pi \sim 1$ and a RP ~ 2. In Fig. 12, the dynamic simulation of a supply chain unit with a stochastic customer demand and a PI controller with $K_j = 0.67$, and $\tau_{I,j} = 3.3$ is shown. The offset is eliminated and the bullwhip is suppressed, but the response of the inventory position is slow. This causes low customer satisfaction during the transient period. Fig. 13 gives the frequency response of $\mathrm{IP}_j(z)/U_{kj}(z)$ with the above PI setting. It is shown that this system is over-damped, and the tracking is hence slow.

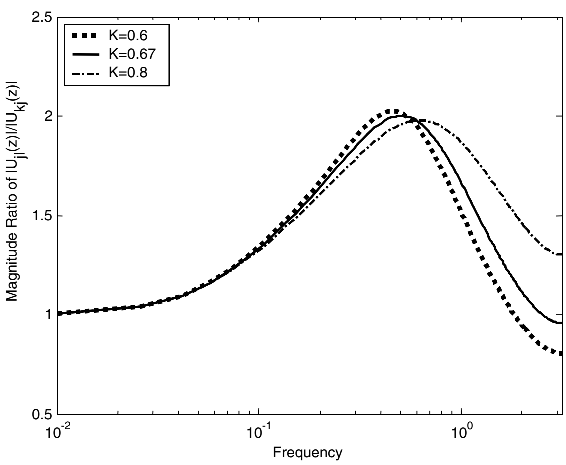

Fig. 10. Frequency responses of $|U_{ji}(z)|/|(U_{kj}(z)|$ with some different K_j values of PI controller and the same value of $\tau_{I,j} = 3.3$.

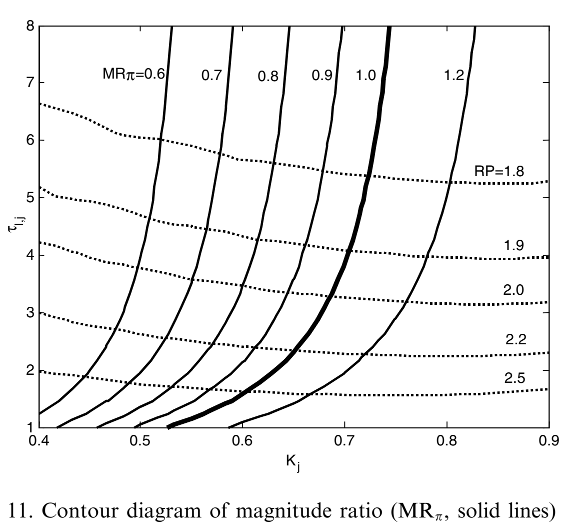

Fig. 11. Contour diagram of magnitude ratio (MR_π, solid lines) and resonant peak (RP, dot lines) as functions of K_j and $\tau_{I,j}$ of a PI controller.

5.3. Cascade control

An obvious alternative to be used is a cascade control scheme as shown in Fig. 14. In the cascade scheme, the set point of the inventory position is raised (or reduced) if the filtered "long term" trend of the difference between the actual inventory position and the demand is less than (or greater than) zero. However, this target is only loosely pursued in the inner loop. The closed loop transfer functions are given by

$$\mathrm{IP}_j(z) = \frac{C_j(z) \times CC_j(z) \times F(z) \times (L+2) - 1}{z - 1 + C_j(z) \times (1 + CC_j(z) \times F(z))} U_{kj}(z) \tag{26}$$

$$U_{ji}(z) = \frac{C_j(z) \times ((L+2) \times CC_j(z) \times F(z) \times (z-1) + CC_j(z) \times F(z) + 1)}{z - 1 + C_j(z) \times (1 + CC_j(z) \times F(z))} \times U_{kj}(z) \tag{27}$$

If an exponential filter with $\alpha = 0.1$ is used for the forecaster $F(z)$, and a medium gain of $K_j = 0.8$ is used for the inner loop, and the following PI cascade controller is used, then

$$CC_j(z) = K_{C,j} \times \left(1 + \frac{1}{\tau_{IC,j}} \frac{z}{z-1}\right) \tag{28}$$

With $\tau_{IC,j} = 5.5$, the Bode plot (Fig. 15) of the closed loop transfer function shows that the bullwhip can be eliminated for high frequency cases with its outer loop gain close to one and inner loop gain slightly lower than one ($K_j = 0.8$). The resonance peak and the bandwidth of this case are much higher than in the pure PI case as shown in Fig. 11. Fig. 16 shows the contour plot of RP and MR_π with respect to $K_{C,j}$ and $\tau_{IC,j}$. The selection space of the outer loop is much wider than in the pure PI case as shown in Fig. 11. Fig. 17 shows that a selection of $K_{C,j} = 1.05$ and $\tau_{IC,j} = 5.5$ for the outer loop and $K_j = 0.8$ for inner loop, works very well without the bullwhip effect. Fig. 13 also shows that $\mathrm{IP}_j(z)/U_{kj}(z)$ is underdamped with a wider bandwidth than in the PI control scheme. Therefore the cascade control scheme results in better tracking of the set point during the transition period. The period with back-order and the magnitude of back-order are both smaller. Hence customer satisfaction is also higher compared with the PI.

5.4. Controller evaluation

In a realistic system, a different cost measure can be imposed by assigning the actual cost of inventory, cost of transportation, cost of order processing, etc. However, results of studies using such cost measures will

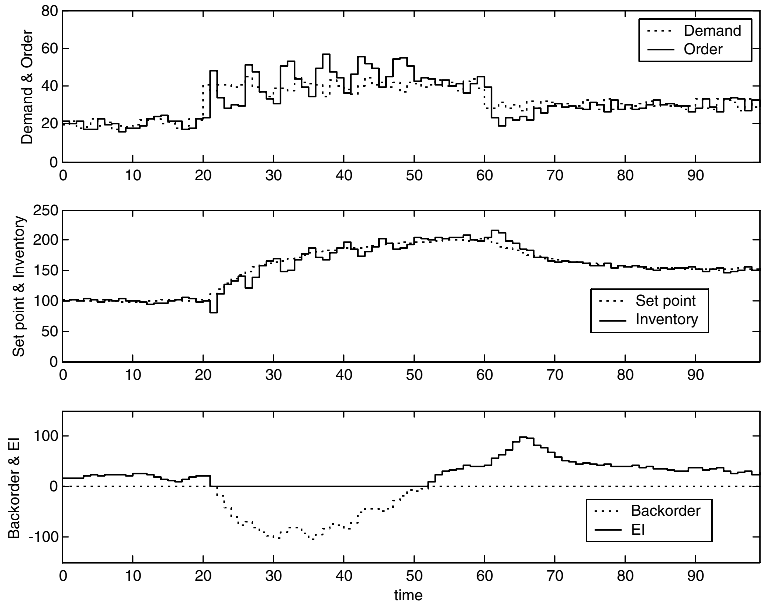

Fig. 12. Dynamic simulation results of a supply chain unit with demand forecasting and a PI controller with $K_j = 0.67$ and $\tau_{I,j} = 3.3$.

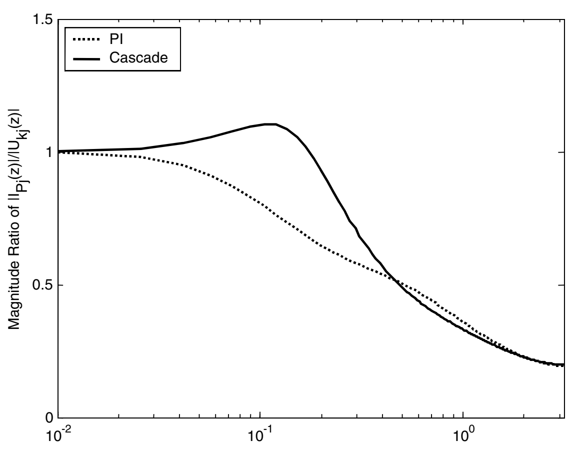

Fig. 13. Frequency responses of $|\mathrm{IP}_j(z)|/|U_{kl}(z)|$ for PI and cascade control.

invariably depend on the cost value actually assigned. Such values vary case by case. Hence, we choose to use more basic indices to evaluate our controller:

1. *Integral absolute error (IAE)*: cumulative difference between actual controlled variable: inventory position and its set point value to evaluate controller performance.
2. *Bullwhip (BW)*: cumulative difference of $|U_{ji}/U_{kj} - 1|$ to indicate the bullwhip effect.
3. *Back-order (BO)*: cumulative difference between inventory and customer demand when there is not enough inventory to indicate customer satisfaction.
4. *Excess inventory (EI)*: cumulative difference between inventory and customer demand when there is more inventory than customers to indicate the cost of inventory.

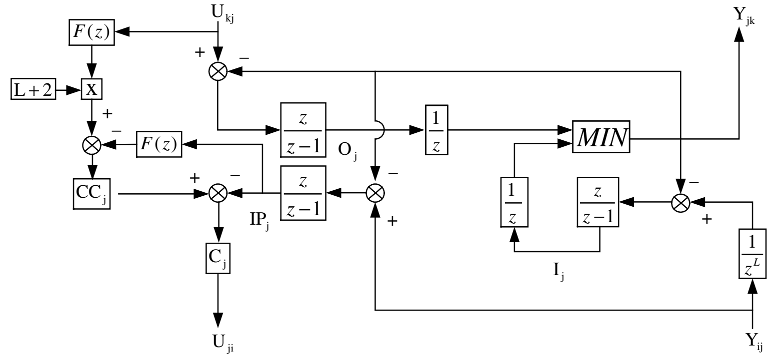

Fig. 14. Cascade control scheme to a supply chain unit.

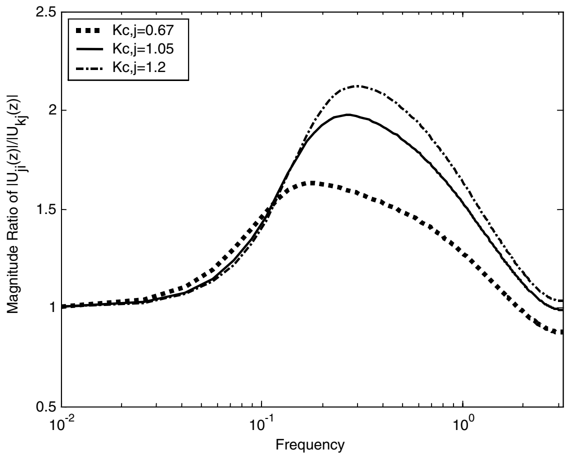

Fig. 15. Frequency responses of $|U_{ji}(z)|/|U_{kj}(z)|$ to a supply chain unit with a cascade controller to its inventory unit with some different $K_{C,j}$ values of the outer-loop PI controller and the same value of reset time $\tau_{IC,j} = 5.5$.

Table 1 listed the four indices obtained by using *P*-control, PI-control and cascade control. While bullwhip can be reduced by detuning the gain of the *P* controller, the IAE and BO deteriorate significantly because *P*-control fails to bring the system to the proper target. PI-control is able to reduce the bullwhip without sacrificing controller performance. Customer satisfaction is improved and excess inventory is also reduced. If a cascade control is used, BW and BO can be further reduced but IAE and ET become slightly inferior than in the PI scheme.

6. Conclusions

The continuous replenishment ordering policy for a distribution node in a supply chain was analyzed using the *z*-transform. Characteristic equations of the closed loop transfer functions are obtained. Stability of the system was investigated. The bullwhip effect is also analyzed. The study proves that the bullwhip effect is inevitable if the standard heuristic ordering policy is employed with demand forecasting. Several alternative ordering policies were formulated as *P*-only, PI and cascade control schemes. Guidelines based on traditional controller tuning methods were provided. By implementing a PI controller, the bullwhip effect of a supply chain unit can be suppressed but long term trends in customer demand can be tracked. The cascade control scheme not only provides efficient control of the inventory position of a supply chain unit without causing the bullwhip effect, but raises the customer satisfaction by providing more active tracking of the customer demand.

While this study investigated the basic dynamic behavior and controller design of a simple model supply chain, there are many possibilities of future works using this approach. Currently, we are looking at how advanced process control methods such as model predictive control can be used. Moreover, while continuous replenishment was practiced, batch ordering is actually more common in most supply chains. Use of the *z*-transform allows us to apply multi-rate sampling techniques to analyze batch ordering strategy. Furthermore, one possible way to reduce bullwhip is to allow the

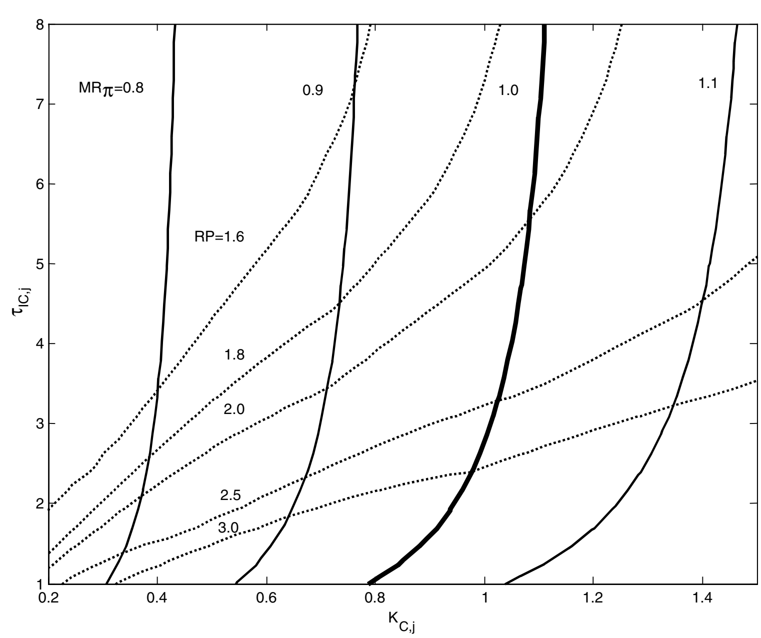

Fig. 16. Contour diagram of magnitude ratio (MR_{π}, solid lines) and resonant peak (RP, dot lines) as functions of $K_{C,j}$ and $\tau_{IC,j}$ of a cascade controller with a *P*-only inner loop controller ($K_j = 0.8$).

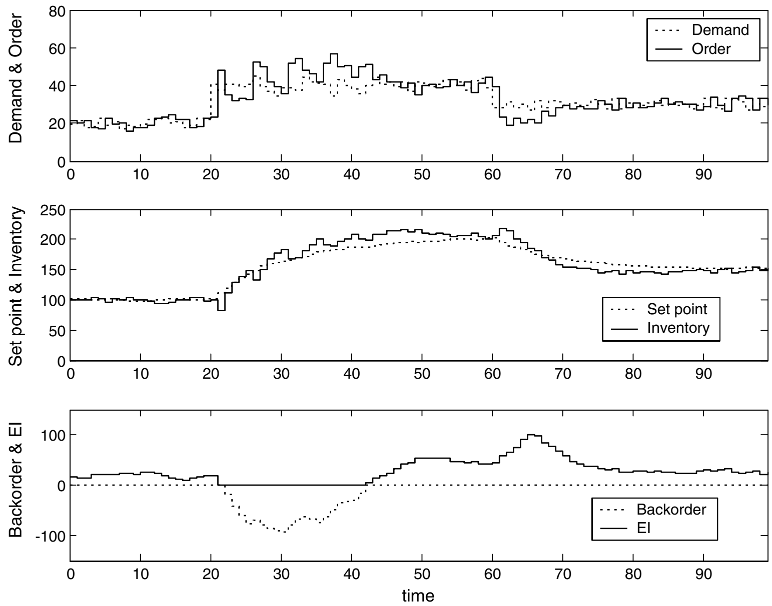

Fig. 17. Dynamic results of cascade control at the outer loop $K_{C,j} = 1.05$, $\tau_{IC,j} = 5.5$, and inner loop control gain $K_j = 0.8$.

Table 1
Performance indices of different control schemes

Control mode	*P*-control		PI control	Cascade control
Parameters	$K_j = 0.7$	$K_j = 1.0$	$K_j = 0.67$, $\tau_{I,j} = 3.3$	$K_j = 0.8$, $K_{C,j} = 1.05$, $\tau_{IC,j} = 5.5$
IAE[a]	3574	2728	454	819
BW[b]	9.68	28.7	9.03	7.91
BO[c]	19,958	11,748	1879	1195
EI[d]	20	20	2138	2379

[a] $\mathrm{IAE} = \int_0^\infty |\mathrm{SP}_j(t) - \mathrm{IP}_j(t)|\,\mathrm{d}t$.
[b] $\mathrm{BW} = \int_0^\infty |U_{ji}(t)/U_{kj}(t) - 1|\,\mathrm{d}t$.
[c] $\mathrm{BO} = \int_0^\infty (Y_{jk}(t) - O_j(t))\,\mathrm{d}t$.
[d] $\mathrm{EI} = \int_0^\infty (I_j(t) - Y_{jk}(t))\,\mathrm{d}t$.

upstream nodes to access customer information several levels downstream, i.e. to make the system more centralized and reduce the number of decision levels. In our case of a single chain network, this would be equivalent to lumping several levels into a single node. The appropriate ordering strategy for remaining nodes would be the same. If there are several customers and the system is completely centralized, there will be no ordering action. The manipulated variables become the delivery commands sent to each node by the centralized decision unit. The system becomes a MIMO system. We can apply the same approach to analyze its behavior. Various complications discussed at the end of Section 2 can also be introduced to make the model more realistic.

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Appendix A

A.1. Derivation of the closed loop transfer function at infinite supply and high stock

If we assume that the supplier has unlimited supply, delivery to node j will be according to what has been ordered:

$$Y_{ij}(z) = z^{-1}O_i(z) = z^{-1}U_{ji}(z) \tag{A.1.1}$$

Moreover, if node j has maintained a high stock level, it will be able to deliver all the customer orders:

$$Y_{jk}(z) = z^{-1}O_j(z) = z^{-1}U_{kj}(z) \tag{A.1.2}$$

Therefore, the inventory position balance (Eq. (8)) can be rewritten as

$$\mathrm{IP}_j(z) = \frac{1}{z-1}(U_{ji}(z) - U_{kj}(z)) \tag{A.1.3}$$

Substituting the ordering policy of node j (Eq. (10)) into the above equation, we get

$$\mathrm{IP}_j(z) = \frac{1}{z-1}(K_j\mathrm{SP}_j(z) - K_j\mathrm{IP}_j(z) - U_{kj}(z)) \tag{A.1.4}$$

Simple rearrangement yields

$$\left(1 + \frac{K_j}{z-1}\right)\mathrm{IP}_j(z) = \frac{1}{z-1}(K_j\mathrm{SP}_j(z) - U_{kj}(z)) \tag{A.1.5}$$

which can be further simplified into Eq. (12).

A.2. Derivation of the closed loop transfer function at infinite supply and low stock

If an upstream supplier has an infinite supply, it will deliver according to the order, as in Eq. (A.1.1). However, if node j keeps a low stock so that there is always less inventory than the amount ordered by the customer, delivery is limited by the inventory:

$$Y_{jk}(z) = z^{-1}I_j(z) \tag{A.2.1}$$

Therefore, substituting Eqs. (A.1.1) and (A.2.1) into the inventory position balance in Eq. (8), we get

$$\mathrm{IP}_j(z) = \frac{1}{z-1}(U_{ji}(z) - I_j(z)) \tag{A.2.2}$$

Using Eqs. (7) and (8), the relation between the actual inventory and the inventory position is given by

$$\begin{aligned} I_j(z) &= \mathrm{IP}_j(z) + \frac{z(z^{-L}-1)}{z-1}Y_{ij}(z) \\ &= \mathrm{IP}_j(z) + \frac{z^{-L}-1}{z-1}U_{ji}(z) \end{aligned} \tag{A.2.3}$$

The amount order is determined by P-control (Eq. (10)); therefore,

$$I_j(z) = \mathrm{IP}_j(z) + \frac{z^{-L}-1}{z-1}K_j(\mathrm{SP}_j(z) - \mathrm{IP}_j(z)) \tag{A.2.4}$$

$$\begin{aligned} \mathrm{IP}_j(z) = \frac{1}{z-1}\Big(&K_j\mathrm{SP}_j(z) - K_j\mathrm{IP}_j(z) - \mathrm{IP}_j(z) \\ &- \frac{z^{-L}-1}{z-1}K_j(\mathrm{SP}_j(z) - \mathrm{IP}_j(z))\Big) \end{aligned} \tag{A.2.5}$$

Simplification and rearrangement give

$$\begin{aligned} \mathrm{IP}_j(z) = \frac{1}{z-1}\Bigg(&\frac{z^{L+1} - z^L + z^L - 1}{(z-1)z^L}K_j\mathrm{SP}_j(z) \\ &- \left(1 + \frac{z^{L+1} - z^L + z^L - 1}{(z-1)z^L}K_j\right)\mathrm{IP}_j(z)\Bigg) \end{aligned} \tag{A.2.6}$$

$$\left(\frac{z}{z-1} + \frac{K_j(z^{L+1}-1)}{(z-1)^2z^L}\right)\mathrm{IP}_j(z) = \frac{z^{L+1}-1}{(z-1)^2z^L}K_j\mathrm{SP}_j(z) \tag{A.2.7}$$

$$\left(1 + \frac{K_j(z^{L+1}-1)}{(z-1)z^{L+1}}\right)\mathrm{IP}_j(z) = \frac{z^{L+1}-1}{(z-1)z^{L+1}}K_j\mathrm{SP}_j(z) \tag{A.2.8}$$

Eliminating the denominator term $(z-1)z^{L+1}$ and rearrangement produce the closed loop transfer function and characteristic equations (14) and (15).

A.3. Derivation of the "Stability" limit of K at infinite supply and low stock

By rearranging Eq. (15) yields

$$z^{L+1} + K_jz^L + K_jz^{L-1} + \cdots K_jz + K_j = 0 \tag{A.3.1}$$

If we assume r_m $(m = 1, 2, 3, \ldots, L+1)$ are the roots of Eq. (A.3.1), then

$$|K_j| = \prod_{m=1}^{m=L+1} |r_m| \tag{A.3.2}$$

From the above Eq. (A.3.2) we know if $K_j > 1$, there must be at least one root in Eq. (A.3.1), whose magnitude is bigger than one.

A.4. Derivation of the closed loop transfer function a limited supply

If the supplier node i has insufficient inventory, then its delivery is limited by its stock level instead of the demand of node j:

$$Y_{ij}(z) = z^{-1}I_i(z) \tag{A.4.1}$$

The inventory position becomes

$$\mathrm{IP}_j(z) = \frac{1}{z-1}(I_i(z) - zY_{jk}(z)) \tag{A.4.2}$$

When node j keeps a high stock as given in Eq. (A.1.2)

$$\mathrm{IP}_j(z) = \frac{1}{z-1}(I_i(z) - U_{kj}(z)) \tag{A.4.3}$$

When node j keeps a low stock

$$\begin{aligned} \mathrm{IP}_j(z) &= \frac{1}{z-1}(I_i(z) - I_j(z)) \\ &= \frac{1}{z-1}\left(I_i(z) - \mathrm{IP}_j(z) - \frac{z(z^{-L}-1)}{z-1}Y_{ij}(z)\right) \end{aligned} \tag{A.4.4}$$

Substituting Eq. (A.3.1) into the above equation and rearrangement give

$$\begin{aligned} \left(1 + \frac{1}{z-1}\right)\mathrm{IP}_j(z) &= \frac{1}{z-1}\left(I_i(z) - \frac{(z^{-L}-1)}{z-1}I_i(z)\right) \\ &= \frac{z^{L+1}-1}{(z-1)^2z^L}I_i(z) \end{aligned} \tag{A.4.5}$$

Further simplification gives

$$\mathrm{IP}_j(z) = \frac{z^{L+1}-1}{(z-1)z^{L+1}}I_i(z) \tag{A.4.6}$$

A.5. Derivation of the "bullwhip" limit of K_j

By taking the square of the norms in Eq. (18), we get

$$\frac{|U_{ji}(z)|^2}{|U_{kj}(z)|^2} = \frac{K_j^2}{(e^{i\omega} + K_j - 1)(e^{-i\omega} + K_j - 1)} \tag{A.5.1}$$

Expanding

$$\begin{aligned}\frac{|U_{ji}(z)|^2}{|U_{kj}(z)|^2} &= \frac{K_j^2}{(1 + 2(K_j - 1)\cos\omega + (K_j - 1)^2)} \\ &= \frac{K_j^2}{(K_j^2 - 2K_j + 2 + 2(K_j - 1)\cos\omega)}\end{aligned} \tag{A.5.2}$$

Therefore,

$$\frac{|U_{ji}(z)|^2}{|U_{kj}(z)|^2} = \frac{K_j^2}{(K_j^2 + 2(\cos\omega - 1)(K_j - 1))} > 1$$
$$\forall\omega \text{ if } K_j > 1 \tag{A.5.3}$$

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